Fully Probabilistic Design of LQ Control with On-Line Parameter Tuning

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Outline:

❶ Intro: Motivation, Purpose, Ideological line

Motivation I.

To design several simple general algorithms for control of class of mechatronic systems (e.g. robotic tools) influenced by uncertain neighborhood interactions both of known and unknown source.

Motivation II.

To understand fundaments of software tools developed in our department.

Starting informational sources:

design and algorithms of adaptive LQ Control >> Josef Böhm

fully probabilistic theory used in control design >> Miroslav Karný

❶ Intro: Motivation, Purpose, Ideological line

Block representation of system and its neighborhood

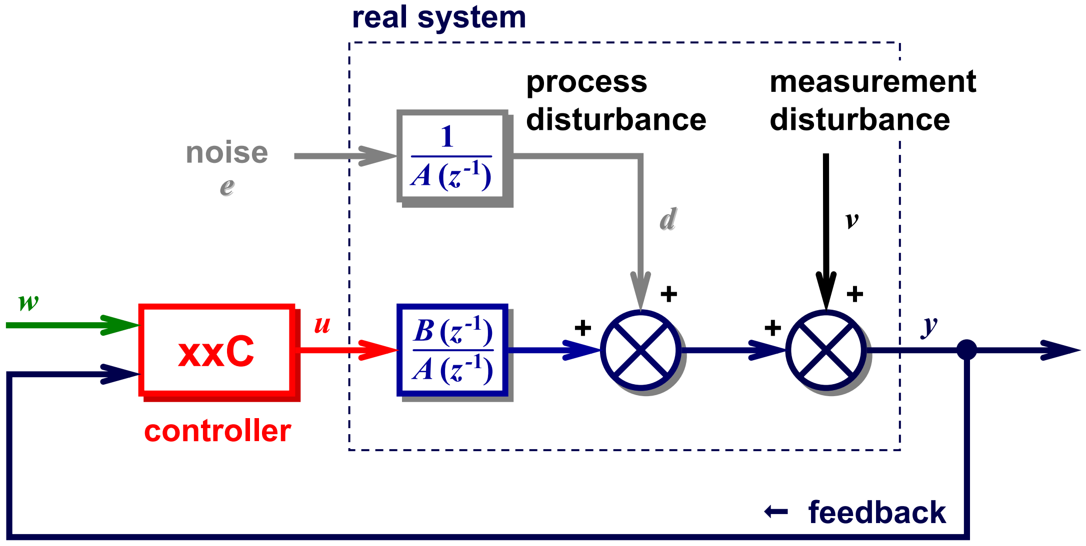


diagram of closed-loop

Main purpose:

To test the different control strategies (LQ Control strategy designed by probabilistic theory is one of them) for selection of suitable control for mechatronic and robotic applications.

❶ Intro: Motivation, Purpose, Ideological line

Theory >> Practice

Theory: Fully Probabilistic Design (FPD) of Control

- general case:

$$\underbrace{J = J(pdf, {}^{I}pdf)}_{} \quad \underbrace{pdf, \; {}^{I}pdf}_{} \qquad \min_{pdf} J \qquad \underbrace{{}^{o}pdf}_{}$$

\>> criterion + model >> minimization >> opt. control

- specific case:

\>> FPD of Control leads to standard LQ Control, i.e. FPD gives new interpretation to LQ Control e.g. a possibility to on-line "fine - tune" or "retune" LQ Controller in view of uncertain neighborhood interactions.

Practice (practical use):

special algorithms for specific FPD need not be designed;
standard LQ Control algorithms can be utilized,
due to probabilistic interpretation, with specific tuning.

❷ FPDofC: Fully Probabilistic Design of Control

Criterion:

$$\mathcal{D}(f \,\|\, {}^{I}\!f) \equiv E\left\{\ln\left(\frac{f(X)}{{}^{I}\!f(X)}\right)\right\} = \int f(X)\ln\left(\frac{f(X)}{{}^{I}\!f(X)}\right)dX$$

Kullback-Leibler divergence (*KL-divergence*) measures proximity of real and ideal distributions of closed-loop

- joint *pdf* representing the real closed-loop behavior:

$$f(X) = f_N \equiv f(\mathbf{x}_{k+N}, u_{k+N-1}, \mathbf{x}_{k+N-1}, u_{k+N-2}, \cdots, u_k, \mathbf{x}_k)$$

assuming $= \left\{\prod_{j=k+1}^{k+N} f(\mathbf{x}_j \mid \mathbf{x}_{j-1}, u_{j-1})\, f(u_{j-1} \mid \mathbf{x}_{j-1})\right\} f(\mathbf{x}_k)$

- joint *pdf* representing the ideal closed-loop behavior:

$${}^{I}\!f(X) = {}^{I}\!f_N \equiv {}^{I}\!f(\mathbf{x}_{k+N}, u_{k+N-1}, \mathbf{x}_{k+N-1}, u_{k+N-2}, \cdots, u_k, \mathbf{x}_k)$$

assuming $= \left\{\prod_{j=k+1}^{k+N} {}^{I}\!f(\mathbf{x}_j \mid \mathbf{x}_{j-1}, u_{j-1})\, {}^{I}\!f(u_{j-1} \mid \mathbf{x}_{j-1})\right\} f(\mathbf{x}_k)$

❷ FPDofC: Fully Probabilistic Design of Control

Task specification:

$$\{{}^{O}f(u_{j-1}\,|\,\mathbf{x}_{j-1})\}_{j=k+1}^{k+N} \in \arg\min_{\{f(u_{j-1}\,|\,\mathbf{x}_{j-1})\}_{j=k+1}^{k+N}} \mathcal{D}(f_N \,||\, {}^{I}f_N)$$

Minimization procedure in brief:

$$\min_{\{f(u_{j-1}\,|\,\mathbf{x}_{j-1})\}_{j=k+1}^{k+N}} \mathcal{D}(f_N \,||\, {}^{I}f_N) = \min_{\{f(u_{j-1}\,|\,\mathbf{x}_{j-1})\}_{j=k+1}^{k+N}} E\sum_{j=k+1}^{k+N} z_j = \cdots$$

dynamic programming

$$= \min_{\{f(u_k|\mathbf{x}_k)\}}\left\{E(z_{k+1}) + \cdots \min_{\{f(u_{k+N-2}|\mathbf{x}_{k+N-2})\}}\left\{E(z_{k+N-1}) + \min_{\{f(u_{k+N-1}|\mathbf{x}_{k+N-1})\}}\{E(z_{k+N})\}\right\}\cdots\right\}$$

$$z_j = \ln\frac{f(\mathbf{x}_j\,|\,\mathbf{x}_{j-1},u_{j-1})\,f(u_{j-1}\,|\,\mathbf{x}_{j-1})}{{}^{I}f(\mathbf{x}_j\,|\,\mathbf{x}_{j-1},u_{j-1})\,{}^{I}f(u_{j-1}\,|\,\mathbf{x}_{j-1})} = j^{th} \text{ partial loss}$$

➾ **Optimal control *pdf***

❷ FPDofC: Fully Probabilistic Design of Control

Optimal control *pdf*

$$ {}^{O}f(u_{j-1} \,|\, \mathbf{x}_{j-1}) = \frac{1}{\gamma(\mathbf{x}_{j-1})} \; {}^{I}f(u_{j-1} \,|\, \mathbf{x}_{j-1}) \; e^{-\delta(u_{j-1}, \mathbf{x}_{j-1})} $$

where factors δ and γ are defined as follows:

$$ \delta(u_{j-1}, \mathbf{x}_{j-1}) \equiv \int f(\mathbf{x}_j \,|\, u_{j-1}, \mathbf{x}_{j-1}) \ln\left(\frac{f(\mathbf{x}_j \,|\, u_{j-1}, \mathbf{x}_{j-1})}{\gamma(\mathbf{x}_j)\; {}^{I}f(\mathbf{x}_j \,|\, u_{j-1}, \mathbf{x}_{j-1})} \right) d\,\mathbf{x}_j $$

$$ \gamma(\mathbf{x}_{j-1}) \equiv \int {}^{I}f\,(u_{j-1} \,|\, \mathbf{x}_{j-1}) \; e^{-\delta(u_{j-1}, \mathbf{x}_{j-1})} \, du_{j-1} $$

The factors are expressed recursively.

They are set up on the basis of minimization procedure.

❸ Model: Probabilistic model representation

$$\mathcal{N}(\mu_y, \sigma_y^2) \equiv \mathcal{N}(\mu_y, r_y): \; f(y) = \frac{1}{\sigma_y\sqrt{2\pi}} e^{-\frac{(y-\mu_y)^2}{2\sigma_y^2}} \equiv \frac{1}{\sqrt{2\pi r_y}} e^{-\frac{(y-\mu_y)^2}{2 r_y}}$$

ARX model:

$$y_k = \underbrace{\sum_{i=1}^{n} b_i u_{k-i} - \sum_{i=1}^{n} a_i y_{k-i}}_{\mu_y} + e_{y_k}, \quad e_{y_k} \sim \mathcal{N}(0, r_y)$$

State-Space model:

$$\mathbf{x}_{k+1} = \mathbf{A}\mathbf{x}_k + \mathbf{B}u_k + \mathbf{e}_{\mathbf{x}_{k+1}}, \qquad \mathbf{e}_{\mathbf{x}_{k+1}} \sim \mathcal{N}(\mathbf{0}, \mathbf{R})$$

$$y_k = \mathbf{C}\mathbf{x}_k$$

$$y_k = \underbrace{\mathbf{C}\mathbf{A}\mathbf{x}_{k-1} + \mathbf{C}\mathbf{B}u_{k-1}}_{\mu_y} + \underbrace{\mathbf{C}\mathbf{e}_{\mathbf{x}_k}}_{e_{y_k}}, \quad e_{y_k} \sim \mathcal{N}(0, r_y) = \mathcal{N}(0, \mathbf{CRC}^T)$$

$$\mathbf{x}_k = \begin{bmatrix} u_{k-1} \\ \vdots \\ u_{k-n+1} \\ y_k \\ \vdots \\ y_{k-n+1} \end{bmatrix} \quad \mathbf{A} = \begin{bmatrix} 0 & \cdots & 0 & 0 & \cdots & 0 \\ 1 & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & \ddots & 0 & 0 & \cdots & 0 \\ b_2 & \cdots & b_n & -a_1 & \cdots & -a_n \\ 0 & \cdots & 0 & 1 & \cdots & 0 \\ 0 & \ddots & 0 & 0 & \ddots & 0 \end{bmatrix} \quad \mathbf{B} = \begin{bmatrix} 1 \\ \vdots \\ 0 \\ b_1 \\ \vdots \\ 0 \end{bmatrix} \quad \mathbf{C} = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ \vdots \\ 0 \end{bmatrix}^T \quad \mathbf{e}_{\mathbf{x}_k} = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ \vdots \\ 0 \end{bmatrix} e_{y_k}$$

❹ ImofPD: Implementation of Probabilistic Design

ImofPD = evaluation:

$$\begin{cases} {}^{O}f(u_{j-1}\,|\,\mathbf{x}_{j-1}) = \dfrac{1}{\gamma(\mathbf{x}_{j-1})}\;{}^{I}f(u_{j-1}\,|\,\mathbf{x}_{j-1})\; e^{-\delta(u_{j-1},\mathbf{x}_{j-1})} \\ \gamma(\mathbf{x}_{j-1}) = fce\Big(f(\mathbf{x}_j\,|\,u_{j-1},\mathbf{x}_{j-1}),\ {}^{I}f(u_{j-1}\,|\,\mathbf{x}_{j-1}),\ \delta(u_{j-1},\mathbf{x}_{j-1})\Big) \\ \delta(u_{j-1},\mathbf{x}_{j-1}) = fce\Big(f(\mathbf{x}_j\,|\,u_{j-1},\mathbf{x}_{j-1}),\ {}^{I}f(\mathbf{x}_j\,|\,u_{j-1},\mathbf{x}_{j-1}),\ \gamma(\mathbf{x}_j)\Big) \end{cases}$$

- *pdf* of the **real** controlled system **state** behavior: $f(\mathbf{x}_j\,|\,u_{j-1},\mathbf{x}_{j-1}) = \cdots$

  \+ system state-space model: $\boldsymbol{\mu}_y = \mathbf{C}\boldsymbol{\mu}_{\mathbf{x}} = \mathbf{C}\mathbf{A}\mathbf{x}_{j-1} + \mathbf{C}\mathbf{B}u_{j-1}$

- *pdf* of the **ideal** controlled system **state** behavior: ${}^{I}f(\mathbf{x}_j\,|\,u_{j-1},\mathbf{x}_{j-1}) = \cdots$

  \+ j^{th} user ideal (i.e. desired values): ${}^{I}y_j = \mathbf{C}\,{}^{I}\boldsymbol{\mu}_{\mathbf{x}} = {}^{I}\mu_y = w_j$

- *pdf* of the **ideal** controlled system **input** behavior: ${}^{I}f(u_{j-1}\,|\,\mathbf{x}_{j-1}) = \cdots$

  \+ value of system input ideal: ${}^{I}u_{j-1} = {}^{I}\mu_u = u_{j-2}$

➯ *pdf* of the **optimal** (designed) controlled system **input** behavior:

$${}^{O}f(u_k\,|\,\mathbf{x}_k) = \cdots,\quad {}^{O}u_k = \cdots$$

4 ImPC: Implementation of Probabilistic Design

- *pdf* of the **real** controlled system **output** behavior:

$$f(\mathbf{Cx}_{k+1} | u_k, \mathbf{x}_k) = \frac{1}{\sqrt{2\pi r_y}} e^{-\frac{1}{2}(\mathbf{Cx}_{k+1} - \mathbf{C}\mu_{\mathbf{x}})^T r_y^{-1} (\mathbf{Cx}_{k+1} - \mathbf{C}\mu_{\mathbf{x}})}$$

- *pdf* of the **ideal** controlled system **output** behavior:

$${}^I f(\mathbf{Cx}_{k+1} | u_k, \mathbf{x}_k) = \frac{1}{\sqrt{2\pi r_y}} e^{-\frac{1}{2}(\mathbf{Cx}_{k+1} - w_{k+1})^T r_y^{-1} (\mathbf{Cx}_{k+1} - w_{k+1})}$$

- *pdf* of the **ideal** controlled system **input** behavior:

$${}^I f(u_k | \mathbf{x}_k) = \frac{1}{\sqrt{2\pi\, {}^I r_u}} e^{-\frac{1}{2}(u_k - u_{k-1})^T r_u^{-1} (u_k - u_{k-1})}$$

➩ *pdf* of the **optimal** (designed) controlled system **input** behavior:

$${}^O f(u_k | \mathbf{x}_k) = \frac{1}{\sqrt{2\pi\, {}^O r_u}} e^{-\frac{1}{2}\, {}^O r_u^{-1} \{u_k + k_{\mathbf{x}}\mathbf{x}_k - \sum_{j=k+1}^{k+N+1} k_{w_j} w_j - k_u u_{k-1}\}^2}$$

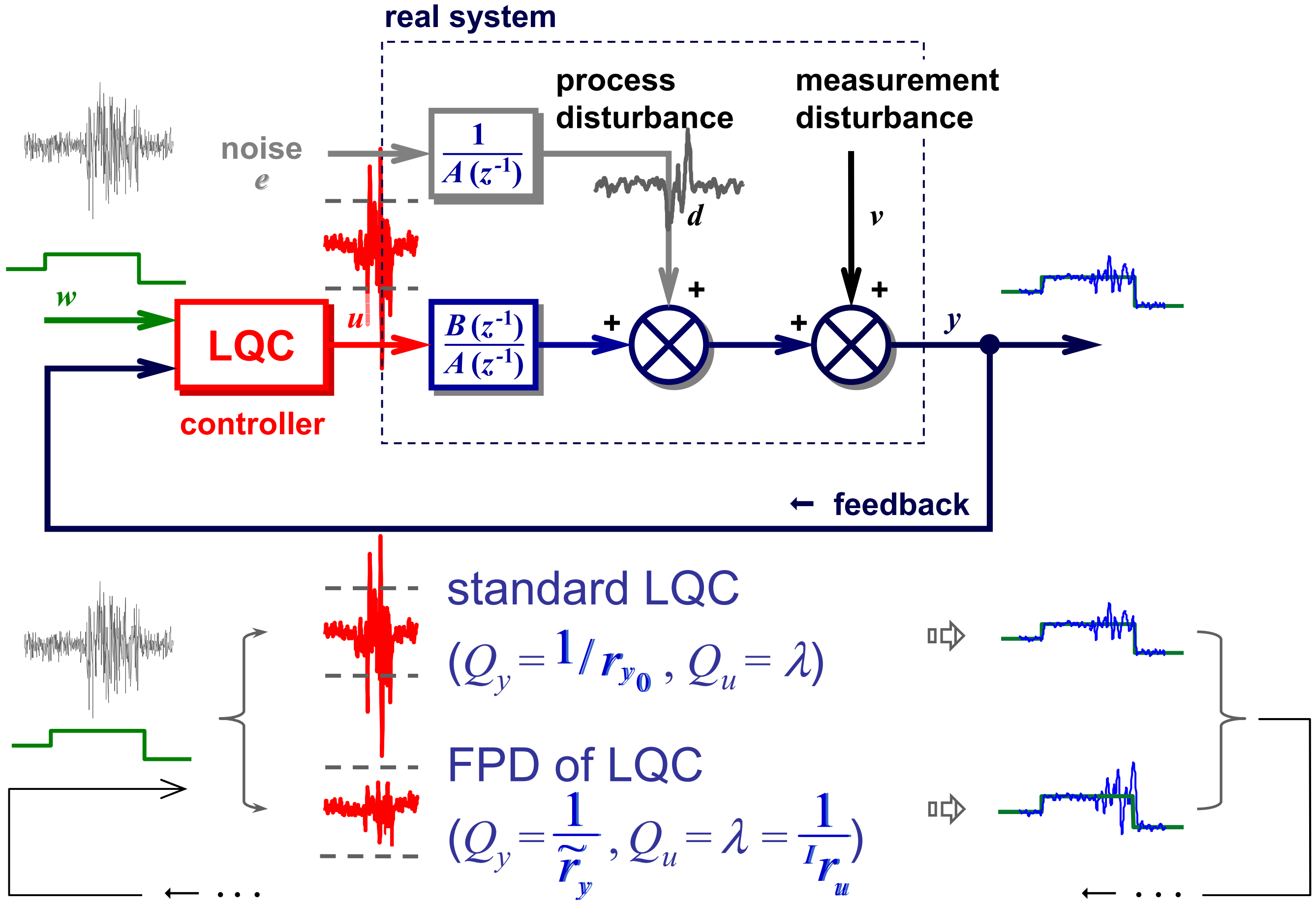
$${}^O u_k = -k_{\mathbf{x}}\mathbf{x}_k + \sum_{j=k+1}^{k+N+1} k_{w_j} w_j + k_u u_{k-1}$$

Note: This result is equivalent to standard LQ Control.

5 Tuning: On-line Fully Probabilistic Tuning of C.

Case specification, in which the on-line tuning is desirable:

Real control of real system in real environment (i.e. real conditions causing interference occurred randomly in control process).



5 Tuning: On-line Fully Probabilistic Tuning of C.

Outline:

- Intro
- FPD of C
- Model
- Im of PD
- **Tuning**
- Exmpl
- Concl.

Quadratic criterion used in LQ Control design:

$$J=\sum_{j=k}^{k+N}\Big((\hat{y}_{j+1}-w_{j+1})^T Q_y\,(\hat{y}_{j+1}-w_{j+1})+(u_j-u_{j-1})^T Q_u\,(u_j-u_{j-1})\Big)$$

standard LQC ($Q_y = 1/r_{y_0}$, $Q_u = \lambda$), FPD of LQC ($Q_y = \dfrac{1}{\tilde{r}_y}$, $Q_u = \lambda = \dfrac{1}{{}^I r_u}$)

where r_y or practically $\hat{r}_{y_i} = \hat{e}_{y_i}\hat{e}_{y_i}^T$, which is very changeable $\Rightarrow$ filtration $\hat{r}_{y_i}$ suitable filter -

- exponential forgetting:

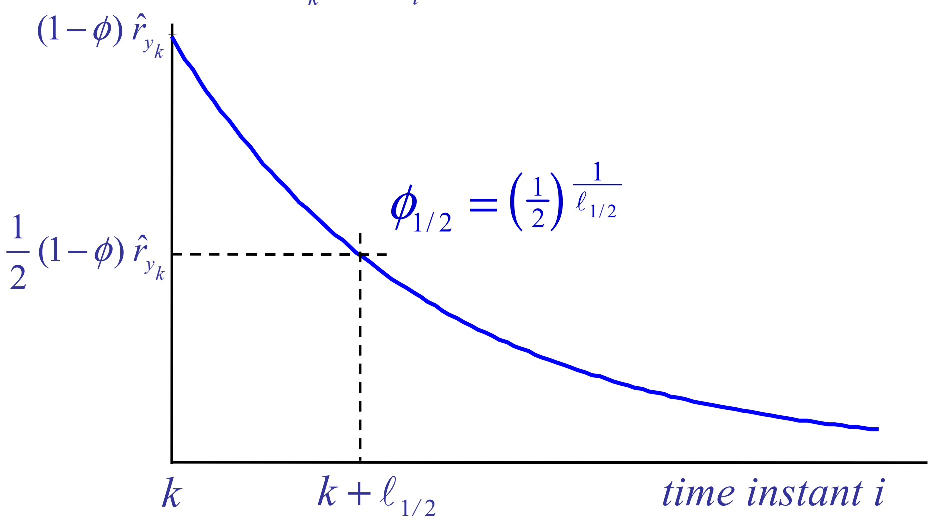
$$\tilde{r}_{y_1} = (1-\phi)\hat{r}_{y_1}$$

$$\tilde{r}_{y_i} = \phi\tilde{r}_{y_{i-1}} + (1-\phi)\hat{r}_{y_i}$$

$$i = 2,\cdots,k$$

$$\tilde{r}_{y_k} = (1-\phi)\sum_{i=1}^{k}\phi^{k-i}\,\hat{r}_{y_i}$$

contribution of $\hat{r}_{y_k}$ to $\tilde{r}_{y_i}$



6 Exmpl: Application examples

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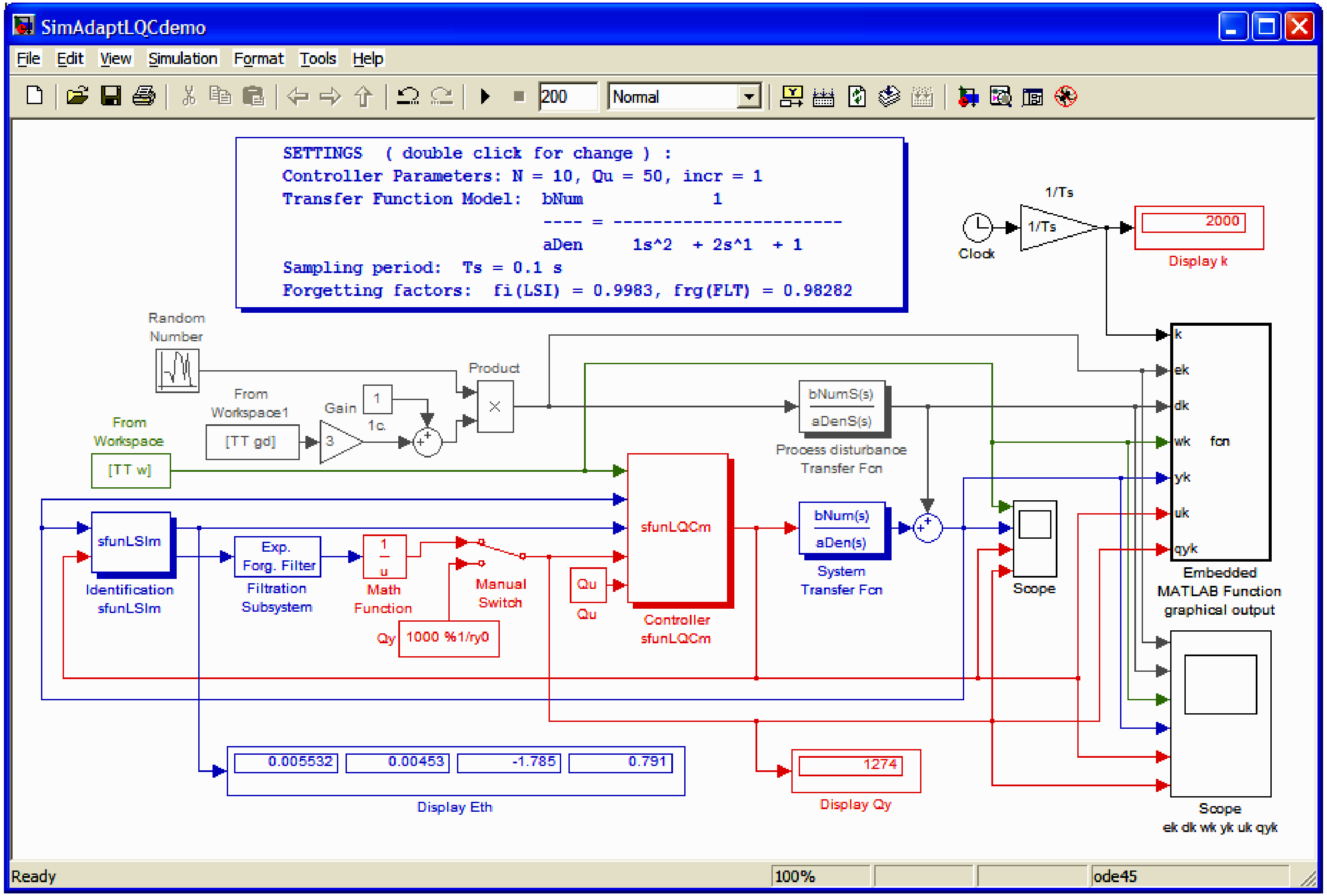


❻ Exmpl: Application examples

Outline:

Intro

FPD∮C

Model

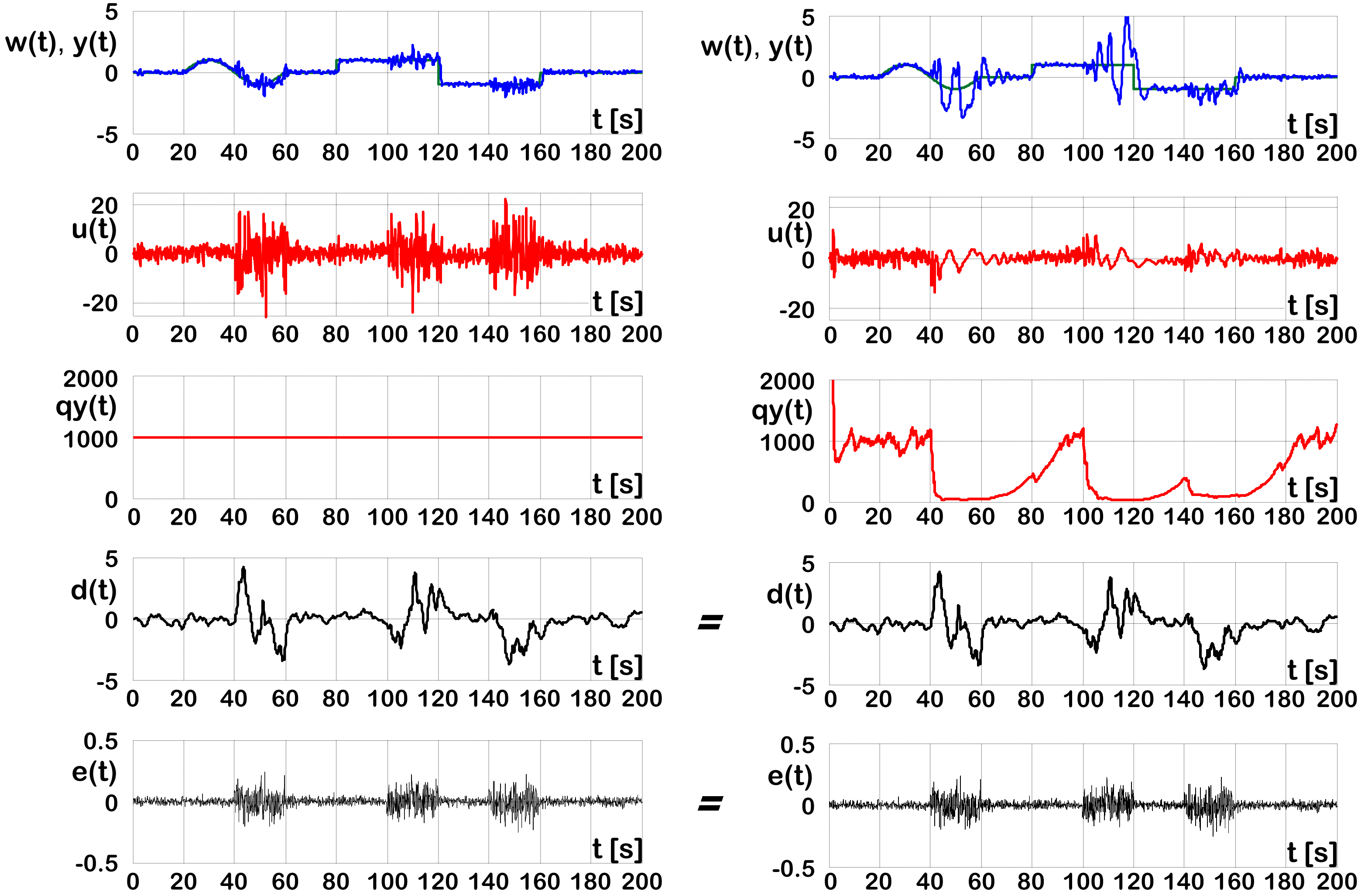
Im∮PD

Tuning

Exmpl

Concl.

standard LQC ($Q_y = 1/r_{y_0}$, $Q_u = \lambda$) FPD of LQC ($Q_y = 1/\tilde{r}_y$, $Q_u = \lambda = 1/{}^{I}r_u$)



7 Concl.: Conclusion

Outline:

Intro

FPD of C

Im of PD

Model

Tuning

Exmpl

Concl.

Several concluding comments

on drawbacks:

- difficult derivation caused by complicated notation (a lot of different parameters/factors in complicated expressions)
- difficult initial parameter/factor determination (various methods of mathematical-physical identification/analysis are needed)
- assumption of specific distributions of the closed-loop variables, which enables this design to be computed, is a limiting factor

on benefits:

- enable designer to use detailed closed-loop description
- fully probabilistic interpretation of LQ Control parameters enabling users to self-fine-tune or retune these parameters
- potential use of parameter fine-tuning or retuning in advanced Generalized Predictive Control strategy

END