# Advances in Bayesian software reliability

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# SOFTWARE RELIABILITY

Software reliability can be defined as *the probability of failure-free operation of a computer code for a specified mission time in a specified input environment*

# SOFTWARE RELIABILITY: MODELS

Seminal paper by Jelinski and Miranda (1972)

More than 100 models after it (Philip Boland, *MMR2002*)

Many models clustered in few classes

Search for unifying models (e.g. Self-exciting process, Chen and Singpurwalla, 1997)

# SOFTWARE RELIABILITY: MODELS

Most software reliability models fall into one of two categories (Singpurwalla and Wilson, 1994)

- [Type I]: models on times between successive failures based on
  - [Type I-1] failure rates (e.g. Jelinski-Moranda)
  - [Type I-2] inter-failure times as function of previous inter-failure times (e.g. random coefficient autoregressive model, Singpurwalla and Soyer, 1985)
- [Type II] models (counting processes) on observed number of failures up to time $t$ (e.g. NHPP)

# SOFTWARE RELIABILITY: MODELS

Failures at $T_1, T_2, \ldots, T_n$

Inter-failure times $T_i - T_{i-1} \sim \mathcal{E}(\lambda_i)$, independent, $i = 1, \ldots, n$

- $\lambda_i = \phi(N - i + 1)$, $\phi \in \mathbb{R}^+, N \in \mathbb{N}$, *(Jelinski-Moranda, 1972)*
  - Program contains an initial number of bugs $N$
  - Each bug contributes the same amount to the failure rate
  - After each observed failure, a bug is detected and corrected

  Straightforward Bayesian inference with priors $N \sim \mathcal{P}(\nu)$ and $\phi \sim \mathcal{G}(\alpha, \beta)$

# SOFTWARE RELIABILITY: MODELS

- $\lambda_i = \phi(N - p(i-1))$, $\phi \in \mathbb{R}^+, N \in \mathbb{N}$, $p \in [0, 1]$, *(Goel-Okumoto, 1978)*
  - Imperfect debugging

- $\lambda_i = \phi\delta^i$, $\phi \in \mathbb{R}^+$, $\delta \in (0, 1)$, *(Moranda, 1975)*
  - Failure rate (geometrically) decreasing

Failure rate constant between failures; different from next model

# SOFTWARE RELIABILITY: MODELS

- $N_t, t \geq 0$ # events by time $t$
- $N(y,s)$ # events in $(y,s]$
- $M(t) = \mathcal{E}N_t$ mean value function
- $M(y,s) = M(s) - M(y)$ expected # events in $(y,s]$

$N_t, t \geq 0$, NHPP with intensity function $\lambda(t)$ *iff*

1. $N_0 = 0$
2. independent increments
3. $\mathcal{P}\{\#\text{ events in } (t, t+h) \geq 2\} = o(h)$
4. $\mathcal{P}\{\#\text{ events in } (t, t+h) = 1\} = \lambda(t)h + o(h)$

$$\Rightarrow \mathcal{P}\{N(y,s) = k\} = \frac{M(y,s)^k}{k!} e^{-M(y,s)}, \forall k \in \mathcal{N}$$

# SOFTWARE RELIABILITY: MODELS

$\lambda(t) \equiv \lambda \; \forall t \Rightarrow$ HPP

- $\lambda(t)$: intensity function of $N_t$

- $\lambda(t) := \lim_{\Delta \to 0} \frac{\mathcal{P}\{N(t, t+\Delta] \geq 1\}}{\Delta}, \; \forall t \geq 0$

- $\mu(t) := \frac{dM(t)}{dt}$: Rocof (rate of occurrence of failures)

Property 3. $\Rightarrow \mu(t) = \lambda(t)$ a.e. $\Rightarrow M(y, s) = \int_y^s \lambda(t)dt$

# SOFTWARE RELIABILITY: ENVIRONMENTS

Techniques to achieve reliable software systems, aimed at

- Fault prevention
- Fault removal
- Fault tolerance (i.e. providing service despite of faults)
- Fault forecasting *(room for statistics ...)*

Michael R. Lyu

Stages, including

- Testing *(e.g. decreasing* $\lambda(t)$ *in NHPP)*
- Operation *(e.g. constant* $\lambda(t)$*)*
- Debugging *(e.g. change in* $\lambda(t)$*)*

# SOFTWARE RELIABILITY: ENVIRONMENTS

Software

- Desktop computing
- Client/server computing
- Web-deployed applications
- .net enterprise (e.g. banking on line)

`Boland, MMR2002`

Intervention

- Perfect repair
- Imperfect repair
- **Bugs introduction**

Different environments, different models

# BAYESIAN SOFTWARE RELIABILITY

*Kuo, 2005*

Review paper

- Models
- Bayesian inference
- Model selection
- Optimal release policy

# OUTLINE

- Statement of the problem
- Hidden Markov model
- Self-exciting process with latent variables
- Change points models
- Future research

# STATEMENT OF THE PROBLEM

Bugs in software induce failures

Fixing current bugs sometimes implies introduction of new bugs

Lack of knowledge about effects of bugs fixing

$\Rightarrow$ need for models allowing for possible, unobserved introduction of new bugs in a context aimed to reduce bugs

# BUGS INTRODUCTION: MODELS

Failures at $T_1, T_2, \ldots, T_n$

Inter-failure times $T_i - T_{i-1} \sim \mathcal{E}(\lambda_i)$, independent, $i = 1, \ldots, n$

- $\lambda_{i+1} = \lambda_i e^{-\theta_i}$, $\lambda_i, \theta_i \in \mathbb{R}^+$, independent
  *(Gaudoin, Lavergne and Soler, 1994)*
  - $\theta_i = 0 \Rightarrow$ no debugging effect
  - $\theta_i > 0 \Rightarrow$ good quality debugging
  - $\theta_i < 0 \Rightarrow$ **bad quality debugging**

# BUGS INTRODUCTION: MODELS

- $\lambda_{i+1} = (1 - \alpha_i - \beta_i)\lambda_i + \mu\beta_i, \qquad \lambda_i, \mu \in \mathbb{R}^+$, *(Gaudoin, 1999)*
  - Imperfect debugging
  - $\alpha_i$ good debugging rate
  - $\beta_i$ **bad debugging rate**

# BUGS INTRODUCTION: MODELS

Birth-death process *(Kremer, 1983)*

- $p_n(t) = \mathcal{P}r\{X(t) = n\}$
- $\nu(t)$ **birth rate**
- $\mu(t)$ death rate
- $a$ initial population

$$p_n^{'}(t) = (n-1)\nu(t)p_{n-1}(t) - n[\nu(t)+\mu(t)]p_n(t) + (n+1)\mu(t)p_{n+1}(t), n \geq 0$$

with $p_{-1} \equiv 0$ and $p_n(0) = \mathbb{1}_{\{n=a\}}$

# HIDDEN MARKOV MODEL

Failure times $t_1 < t_2 < \ldots < t_n$ in $(0, y]$

$Y_t$ latent process describing *reliability status* of software at time $t$ (e.g. growing, decreasing and constant)

$Y_t$ may change only after a failure
$\Rightarrow Y_t = Y_m$ for $t \in (t_{m-1}, t_m]$, $m = 1, \ldots, n+1$
with $t_0 = 0$, $t_{n+1} = y$ and $Y_{t_0} = Y_0$ *given for now*
$\Rightarrow$ consider $\{Y_n\}_{n \in \mathbb{N}}$ Markov chain with discrete state space $E$

$X_m$ interarrival time of $m$-th failure, $m = 1, \ldots, n$

# HIDDEN MARKOV MODEL

Markov chain $Y = \{Y_n\}_{n \in \mathbb{N}}$

- $E$ discrete state space *(card*$(E) = k < \infty$*)*
- $\mathbb{P}$ transition matrix with rows $\mathbb{P}_i = (P_{i1}, \ldots, P_{ik})$, $i = 1, \ldots, k$

Interarrival times $X_m | Y_m = i \sim \mathcal{E}(\lambda(i)), i = 1, \ldots, k, m = 1, \ldots, n$

$\mathbb{P}$ and $\lambda(i)$ unknown

# HIDDEN MARKOV MODEL

*(Durand and Gaudoin, 2005)*

## Parameter estimation

- Data partially observed (only $X_m$ but not $Y_m$)
  $\Rightarrow$ difficult parameter estimation by maximum likelihood

- $\Rightarrow$ EM algorithm for likelihood maximisation in the context of missing values *(McLachlan and Krishnan, 1997)*

- $\Rightarrow$ sequence of values converging to the consistent solution of the likelihood equation, **provided** the starting point is close to the optimal point

- $\Rightarrow$ start from many initial values

# HIDDEN MARKOV MODEL

*(Durand and Gaudoin, 2005)*

Hidden states number estimation

- Take any $k \in [K_{\text{min}}, K_{\text{max}}]$
- For each $k$ compute MLE via EM algorithm
- Choose $k$ with lowest BIC

# HIDDEN MARKOV MODEL

*(Durand and Gaudoin, 2005)*

Selection of transition matrix via BIC

E.g. ordered $\lambda(1) > \lambda(2) > \ldots > \lambda(k)$

- Upper triangular matrix $\Rightarrow$ failure rates can only decrease
- Tridiagonal matrix $\Rightarrow$ only *small* increase and decrease in failure rate

# HIDDEN MARKOV MODEL

$X_m$'s independent given $Y \Rightarrow f(X_1, \ldots, X_n|Y) = \prod_{m=1}^{n} f(X_m|Y)$

$\mathbb{P}_i \sim \mathcal{D}ir(\alpha_{i1}, \ldots, \alpha_{ik}), \forall i \in E$, i.e. $\pi(\mathbb{P}_i) \propto \prod_{j=1}^{k} P_{ij}^{\alpha_{ij}-1}$

Independent $\lambda(i) \sim \mathcal{G}(a(i), b(i)), \forall i \in E$

Interest in posterior distribution of $\Theta = (\lambda^{(k)}, \mathbb{P}, Y^{(n)})$

- $\lambda^{(k)} = (\lambda(1), \ldots, \lambda(k))$

- $Y^{(n)} = (Y_1, \ldots, Y_n)$

# LIKELIHOOD

*For observed* $Y$, likelihood given by

$$\begin{aligned} f(X_1,\ldots,X_n,Y_1,\ldots,Y_n) &= f(X_1,\ldots,X_n|Y_1,\ldots,Y_n)f(Y_1,\ldots,Y_n) \\ &= \prod_{m=1}^{n} P_{Y_{m-1}Y_m}\lambda(Y_m)e^{-\lambda(Y_m)X_m} \end{aligned}$$

Here unobserved $Y$ treated as *parameter*

$$\Rightarrow L(\Theta) = \prod_{m=1}^{n} P_{Y_{m-1}Y_m}\lambda(Y_m)e^{-\lambda(Y_m)X_m}$$

Posterior distribution $\pi(\Theta|X_1,\ldots,X_n,Y_1,\ldots,Y_n)$ proportional to

$$\prod_{m=1}^{n}\left[P_{Y_{m-1}Y_m}\lambda(Y_m)e^{-\lambda(Y_m)X_m}\right]\cdot\prod_{i=1}^{k}\left[[\lambda(i)]^{a(i)-1}e^{-b(i)\lambda(i)}\prod_{j=1}^{k}P_{ij}^{\alpha_{ij}-1}\right]$$

# FULL CONDITIONAL POSTERIORS

- $\mathbb{P}_i | Y^{(n)} \sim \mathcal{D}ir(\alpha_{ij} + \sum_{m=1}^{n} \mathbf{1}_{\{Y_{m-1}=i, Y_m=j\}}; j \in E), \forall i \in E$

- $\lambda(i) | Y^{(n)}, X^{(n)} \sim \mathcal{G}(a^*(i), b^*(i)), \forall i \in E$

  ⋆ $a^*(i) = a(i) + \sum_{m=1}^{n} \mathbf{1}_{\{Y_m=i\}}$ & $b^*(i) = b(i) + \sum_{m=1}^{n} \mathbf{1}_{\{Y_m=i\}} X_m$

  ⋆ $X^{(n)} = (X_1, \ldots, X_n)$

- $\pi(Y_m | Y^{(-m)}, \lambda(Y_m), X^{(n)}, \mathbb{P}) \propto P_{Y_{m-1}, Y_m} \lambda(Y_m) e^{-\lambda(Y_m) X_m} P_{Y_m, Y_{m+1}}$

  ⋆ $\sum_{j \in E} P_{Y_{m-1}, j} \lambda(j) e^{-\lambda(j) X_m} P_{j, Y_{m+1}}$ normalizing constant

  ⋆ $Y^{(-m)} = (Y_1, \ldots, Y_{m-1}, Y_{m+1}, \ldots, Y_n)$

# POSTERIOR SAMPLE AND QUANTITIES

Gibbs sampling: posterior sample from $\pi(\Theta|X^{(n)})$ by iteratively drawing from the given full conditional posterior distributions

Posterior predictive distribution of $X_{n+1}$ after observing $X^{(n)}$

$$\pi(X_{n+1}|X^{(n)}) = \sum_{j \in E} \int \pi(X_{n+1}|\lambda(j)) P_{Y_n,j} \pi(\Theta|X^{(n)}) d\Theta,$$

approximated as a Monte Carlo integral via

$$\pi(X_{n+1}|X^{(n)}) \approx \frac{1}{G} \sum_{g=1}^{G} \pi(X_{n+1}|\lambda^g(Y_{n+1}^g))$$

with $Y_{n+1}^g$ sampled, given the posterior sample $Y_n^g$, using the Dirichlet posterior on $\mathbb{P}_{Y_n^g}$

# POSSIBLE EXTENSIONS

Selection of number of hidden states via Reversible Jump MCMC *(Green, 1995)* $\Rightarrow$ allows for simulation of posterior distributions in parameter spaces of variable size

Ordered $\lambda(1) > \lambda(2) > \ldots > \lambda(k)$

RJMCMC with steps

- [Move] $\lambda_i$ changed to another value in $(\lambda_{i-1}, \lambda_{i+1})$
- [Death] Merge $\lambda_i$ and $\lambda_{i+1}$ into $\lambda_i^*$ and rearrange indices
- [Birth] Split $\lambda_i$ into $\lambda_{i,1}$ and $\lambda_{i,2}$ and rearrange indices

# POSSIBLE EXTENSIONS

- Prior for $Y_0$
- Dynamic models for $\lambda$
- Nonhomogeneous Markov chain
- Estimation of stationary distribution (?)

# JELINSKI-MORANDA DATA

34 software failure times

2 states for $Y_m$

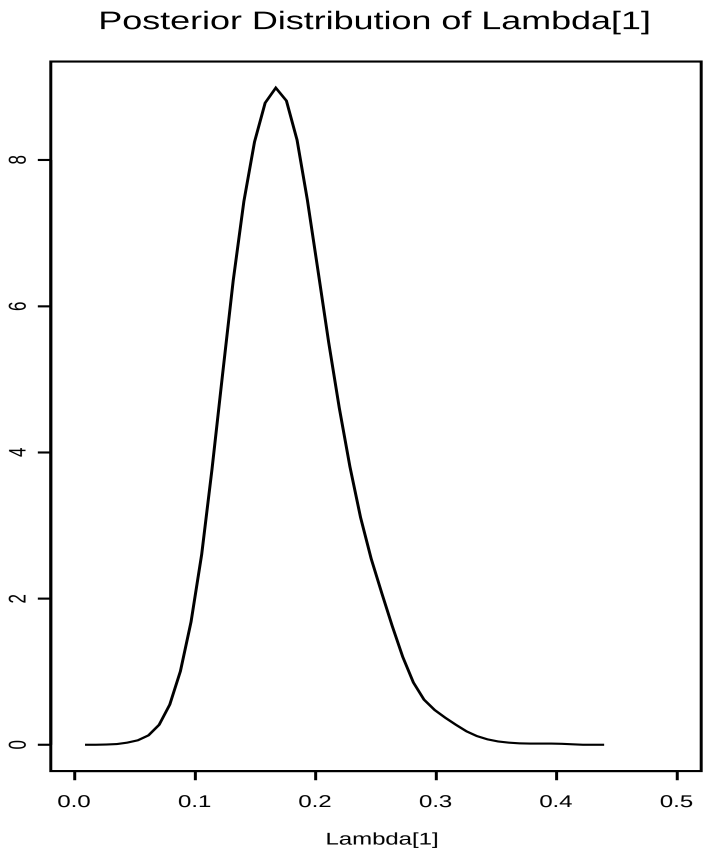
$\mathbb{P}_i \sim \mathcal{B}eta(1,1), i = 1,2$ *(uniform)*

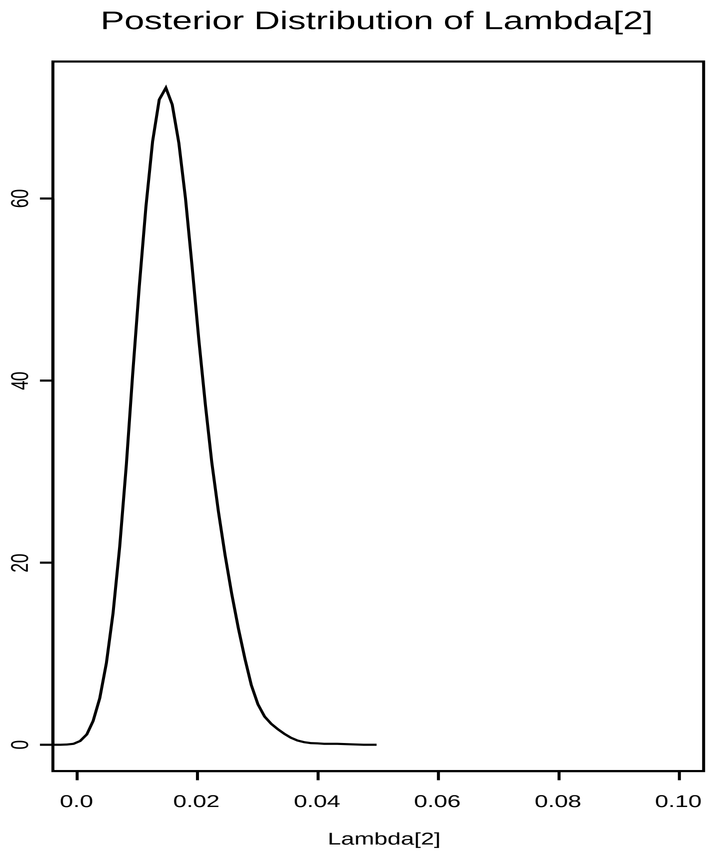
$\lambda(i) \sim \mathcal{G}(0.01, 0.01), i = 1,2$ *(diffuse)*

5000 iterations

Convergence of Gibbs sampler pretty good

# JELINSKI-MORANDA DATA



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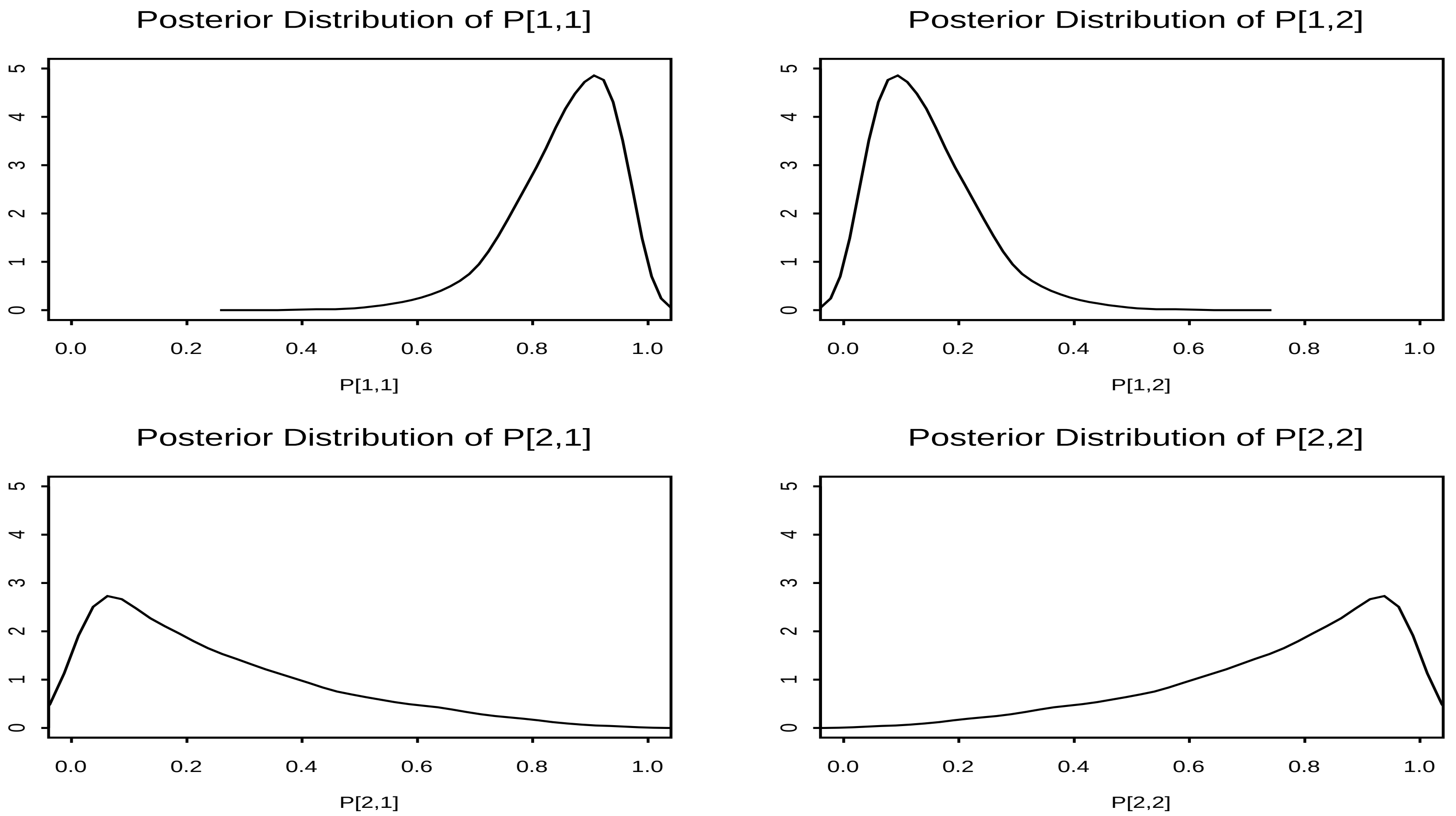

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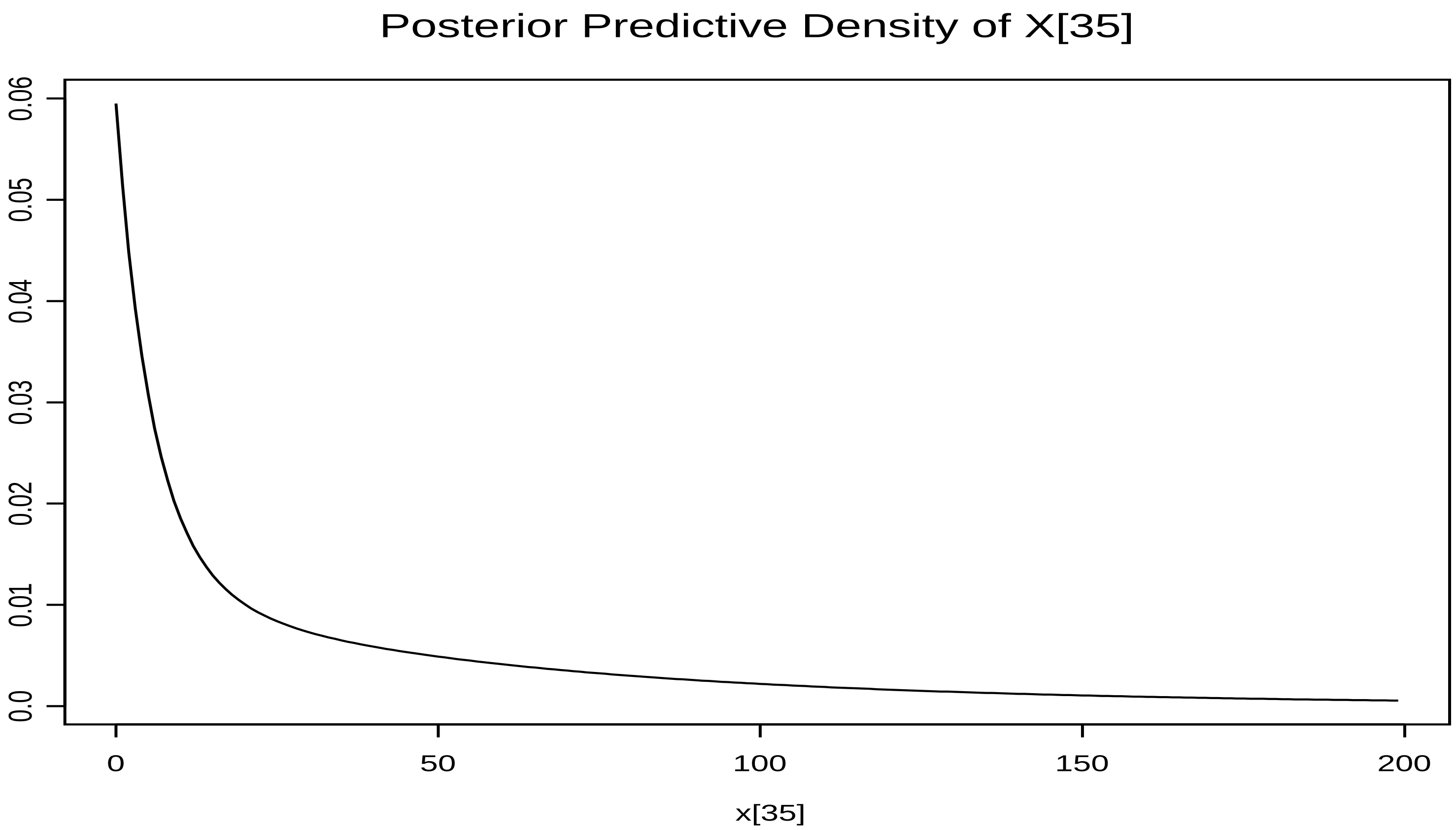

# JELINSKI-MORANDA DATA

Posterior Probabilities of State 1 over Time

| $m$ | $X_m$ | $P(Y_m = 1\|D)$ | $m$ | $X_m$ | $P(Y_m = 1\|D)$ | $m$ | $X_m$ | $P(Y_m = 1\|D)$ |
|---|---|---|---|---|---|---|---|---|
| 1 | 9 | 0.8486 | 2 | 12 | 0.8846 | 3 | 11 | 0.9272 |
| 4 | 4 | 0.9740 | 5 | 7 | 0.9792 | 6 | 2 | 0.9874 |
| 7 | 5 | 0.9810 | 8 | 8 | 0.9706 | 9 | 5 | 0.9790 |
| 10 | 7 | 0.9790 | 11 | 1 | 0.9868 | 12 | 6 | 0.9812 |
| 13 | 1 | 0.9872 | 14 | 9 | 0.9696 | 15 | 4 | 0.9850 |
| 16 | 1 | 0.9900 | 17 | 3 | 0.9886 | 18 | 3 | 0.9858 |
| 19 | 6 | 0.9714 | 20 | 1 | 0.9584 | 21 | 11 | 0.7100 |
| 22 | 33 | 0.2036 | 23 | 7 | 0.3318 | 24 | 91 | 0.0018 |
| 25 | 2 | 0.6012 | 26 | 1 | 0.6104 | 27 | 87 | 0.0020 |
| 28 | 47 | 0.0202 | 29 | 12 | 0.2788 | 30 | 9 | 0.2994 |
| 31 | 135 | 0.0006 | 32 | 258 | 0.0002 | 33 | 16 | 0.1464 |
| 34 | 35 | 0.0794 | | | | | | |

*Expected posterior probability of the "bad" state decreases as we observe longer failure times*

# MUSA SYSTEM 1 DATA

136 software failure times

2 states for $Y_m$

$\mathbb{P}_i \sim \mathcal{B}eta(1,1), i = 1,2$ *(uniform)*

$\lambda(i) \sim \mathcal{G}(0.01, 0.01), i = 1,2$ *(diffuse)*

5000 iterations

Convergence of Gibbs sampler pretty good

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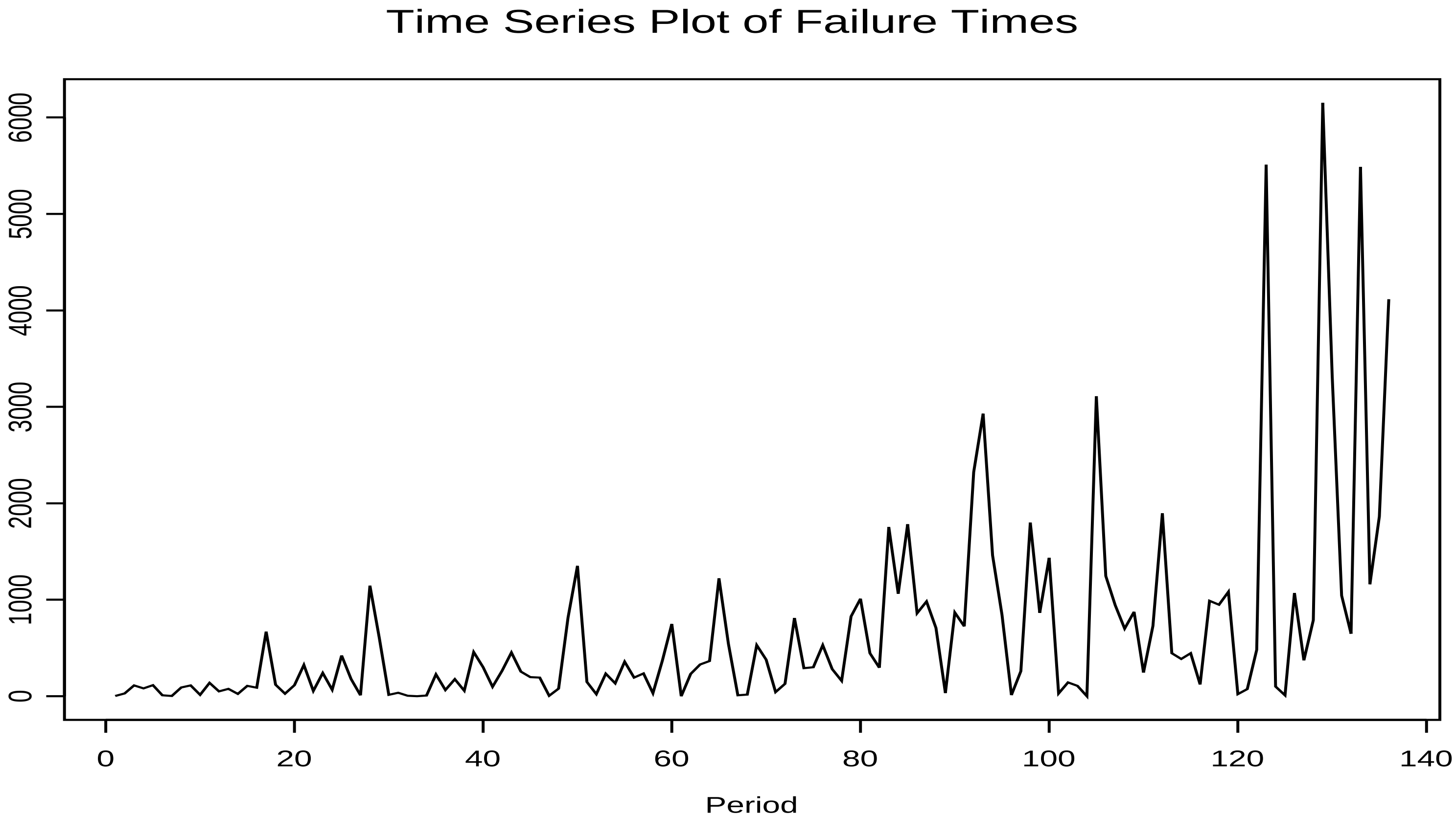

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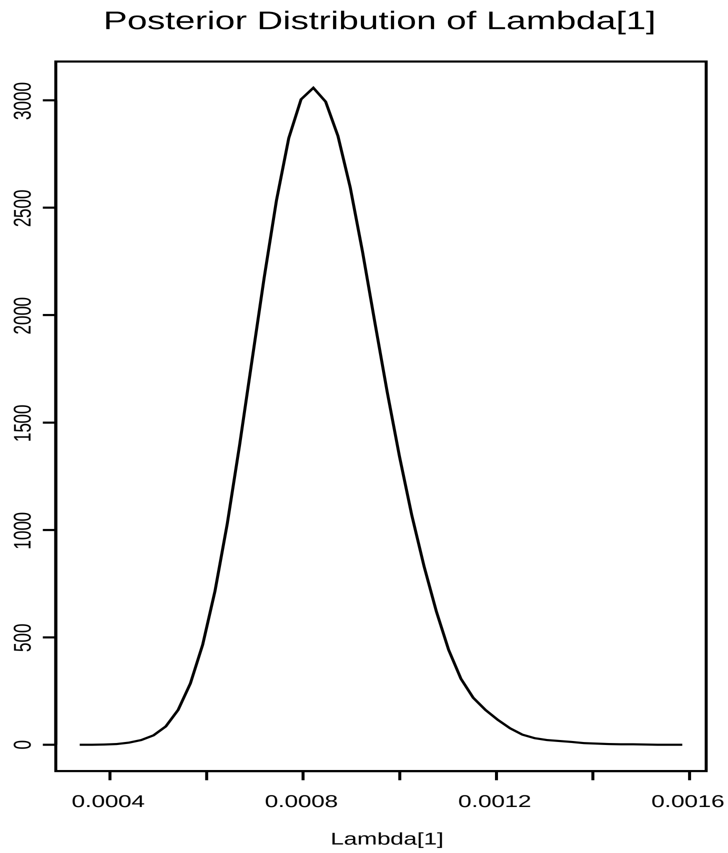


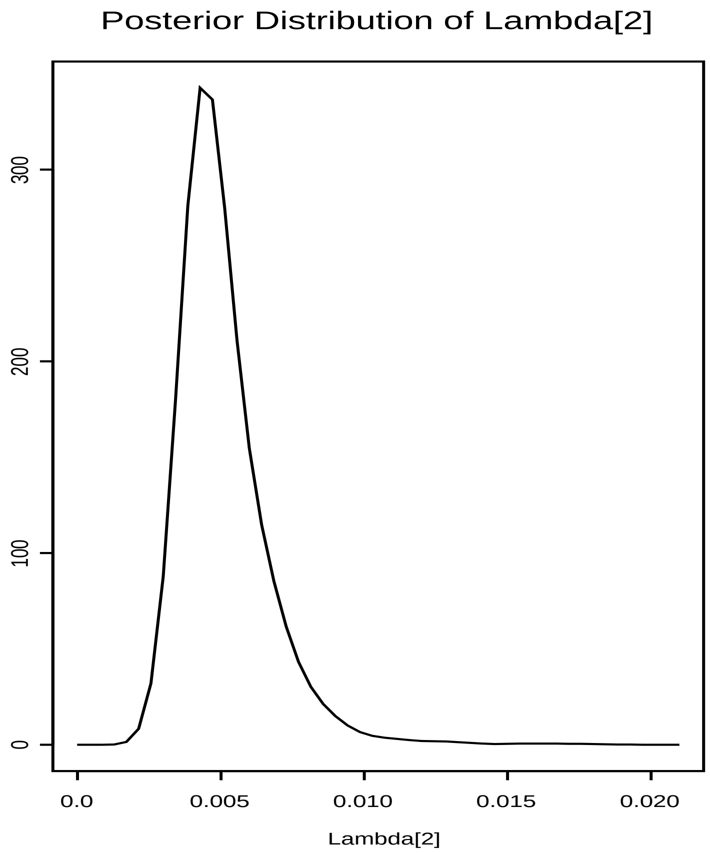

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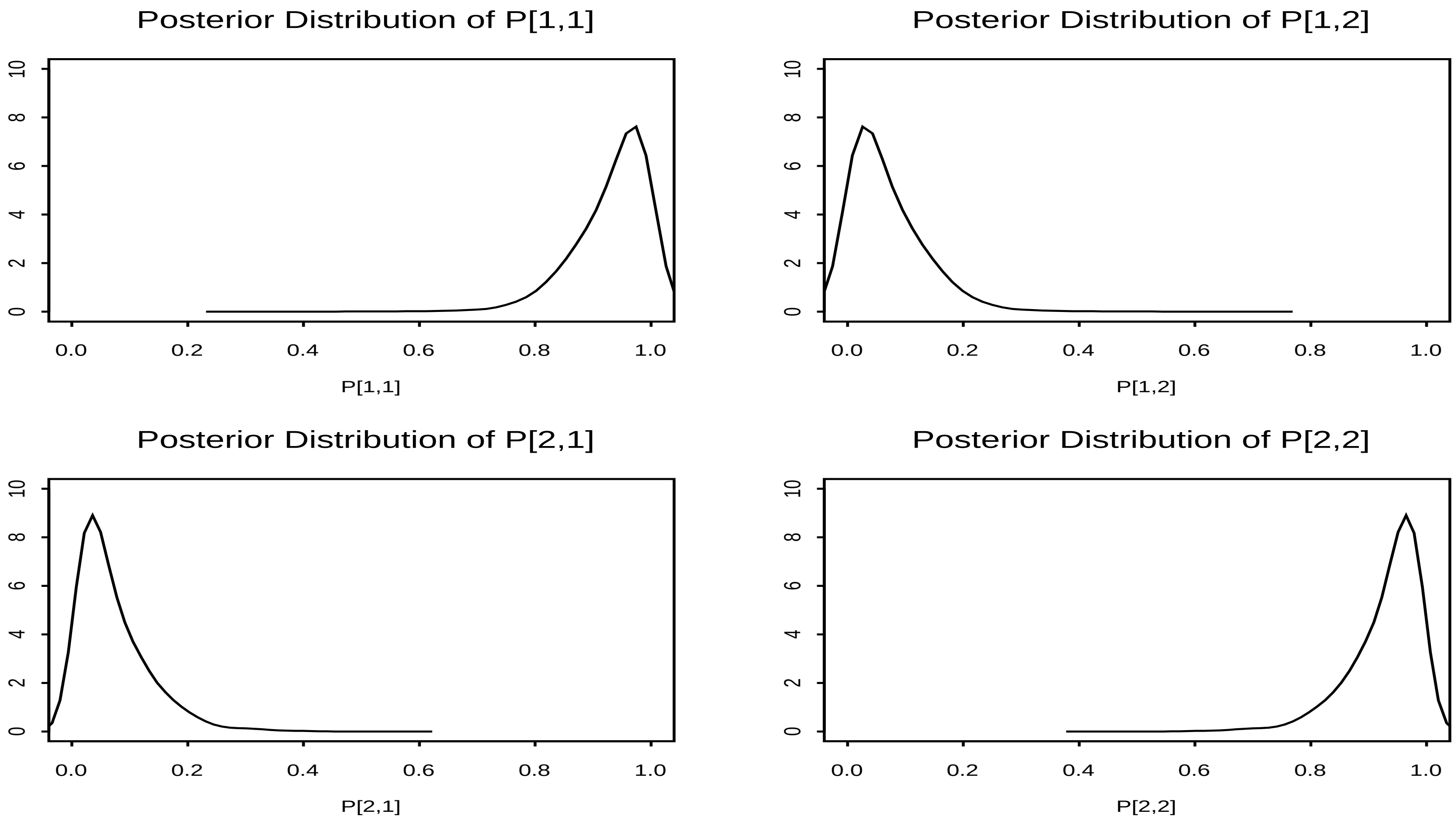

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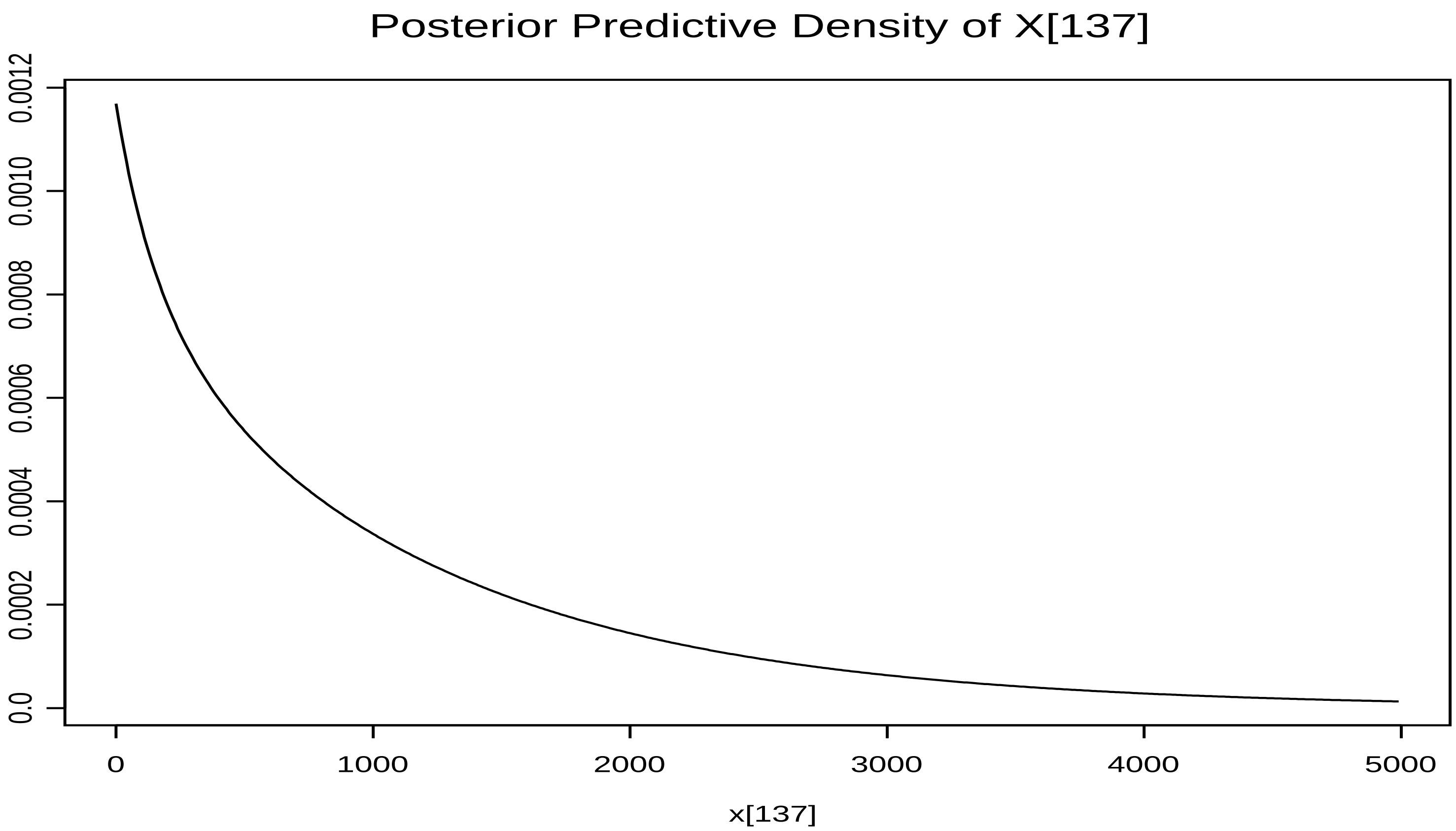

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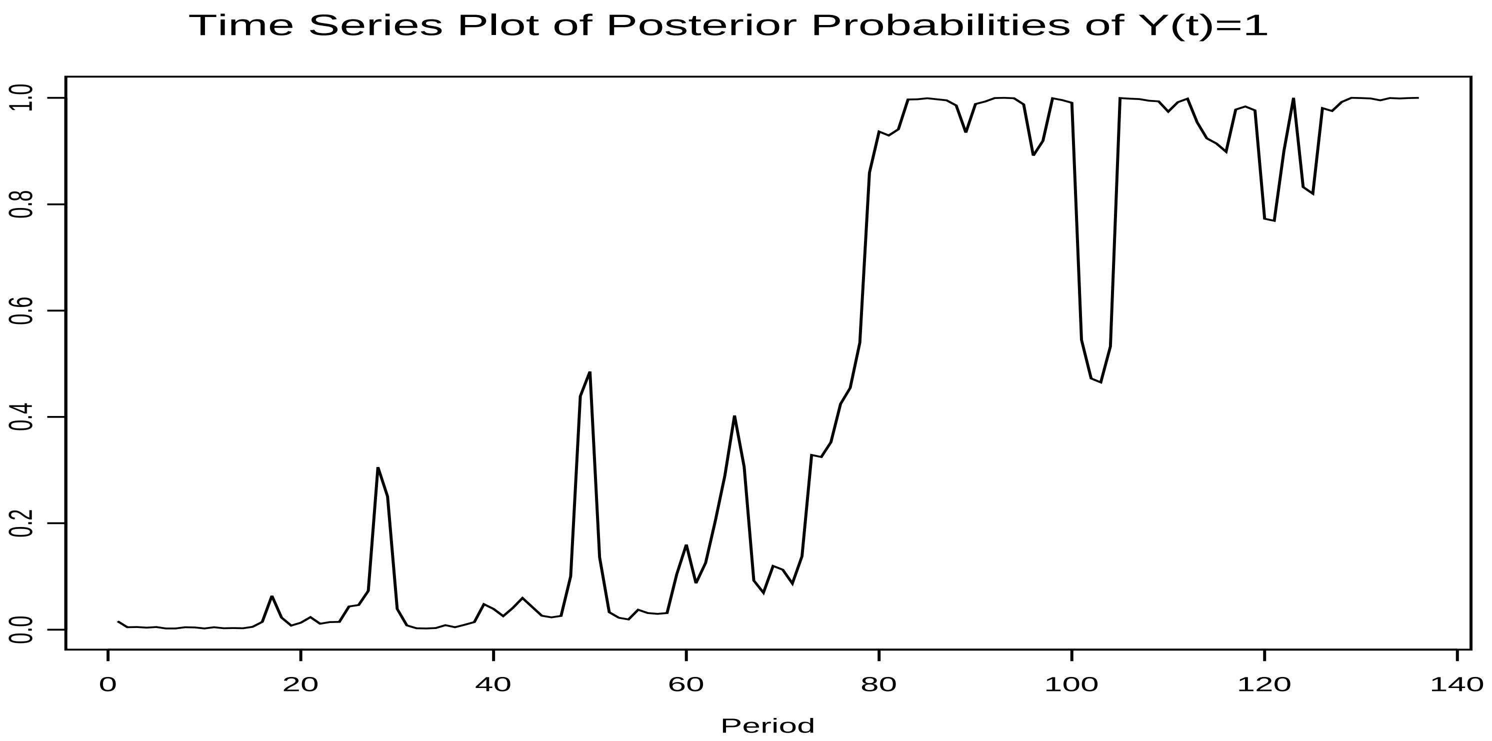


*Expected posterior probability of the "good" state increases as we observe longer failure times*

# SELF-EXCITING PROCESS WITH LATENT VARIABLES

NHPPs widely used in (software) reliability, characterised by an intensity function $\mu(t)$

Self-exciting processes (SEPs) add extra terms $g(t - t_i)$ to the intensity as a consequence of events at $t_i$ (e.g. introduction of new bugs)

Binary latent variables modelling the introduction of new bugs

$\Rightarrow$ SEP with intensity $\lambda(t) = \mu(t) + \sum_{j=1}^{N(t^-)} Z_j g_j(t - t_j)$

- $\mu(t)$ intensity of process w/o introduction of new bugs
- $N(t^-)$ number of failures right before $t$
- $t_1 < t_2 < \ldots < t_n$ failures in $(0, T]$
- $Z_j = 1$ if bug introduced after $j$–th failure and $Z_j = 0$ o.w.
- $g_j(u) \geq 0$ for $u > 0$ and $= 0$ o.w.

# SELF-EXCITING PROCESS: LIKELIHOOD

$\underline{t} = (t_1, \ldots, t_n)$ failures in $(0, T]$

$\underline{Z} = (Z_1, \ldots, Z_n)$ latent variables at $\underline{t} = (t_1, \ldots, t_n)$

Likelihood $L(\theta; \underline{t}, \underline{Z}) = f(\underline{t}|\underline{Z}, \theta) f(\underline{Z}|\theta)$

$$
\begin{aligned}
f(\underline{t}|\underline{Z}, \theta) &= \prod_{i=1}^{n} \lambda(t_i) e^{-\int_0^T \lambda(t) dt} \\
&= \prod_{i=1}^{n} \left[ \mu(t_i) + \sum_{j=1}^{i-1} Z_j g(t_i - t_j) \right] e^{-\int_0^T \mu(t) dt - \sum_{j=1}^{N(T^-)} Z_j \int_0^{T-t_j} g_j(t) dt}
\end{aligned}
$$

*[$\theta$ omitted]*

# SELF-EXCITING PROCESS: ASSUMPTIONS

PLP $\mu(t) = M\beta t^{\beta-1}$, $M > 0, \beta > 0$ & $\mu \equiv g_j, \forall j$

Other possibilities *(still to be explored)*

- $(M_i, \beta_i) \neq (M_j, \beta_j), i \neq j$ & $(M_j, \beta_j) \neq (M_0, \beta_0)[i.e.\ \mu(t)], \forall j$

- $M_0 > M_1 > \ldots > M_n,$, e.g. $M_j = M_0\gamma^j, 0 < \gamma < 1$

- $M_j = \alpha M_{j-1} + \delta, 0 < \alpha < 1, \delta > -\alpha M_{j-1}$

- $\mu(t) = M\beta e^{-\beta t}$, with $\int_0^\infty \mu(dt) = M < \infty$

# SELF-EXCITING PROCESS: LIKELIHOOD

$$
\begin{aligned}
f(\underline{t}|\underline{Z},\theta) &= M^n\beta^n\prod_{i=1}^{n}\left[t_i^{\beta-1}+\sum_{j=1}^{i-1}Z_j(t_i-t_j)\right]e^{-M\left[T^\beta+\sum_{j=1}^{N(T^-)}Z_j(T-t_j)^\beta\right]} \\
&= M^n\beta^n\prod_{i=1}^{n}A_i(\beta,Z^{(i-1)})e^{-MB(\beta,Z^{(n)})}
\end{aligned}
$$

- $Z^{(i)}=(Z_1,\ldots,Z_i)$
- $A_i(\beta,Z^{(i-1)})=t_i^{\beta-1}+\sum_{j=1}^{i-1}Z_j(t_i-t_j)$
- $B(\beta,Z^{(n)})=T^\beta+\sum_{j=1}^{N(T^-)}Z_j(T-t_j)^\beta$

# SELF-EXCITING PROCESS: LIKELIHOOD

$Z_j \sim \mathcal{B}ern(p_j), \forall j$

$$f(\underline{t}, \underline{Z}|\theta) = f(\underline{t}|\underline{Z}, \theta) f(\underline{Z}|\theta) = f(\underline{t}|\underline{Z}, \theta) \prod_{i=1}^{n} p_j^{Z_j} (1 - p_j)^{1 - Z_j}$$

Two possibilities

- Sum over all $Z^{(n)} \Rightarrow f(\underline{t}|\theta)$
- Treat $Z_j$'s as *parameters* and look for full conditionals (for MCMC) *[we follow this]*

# SELF-EXCITING PROCESS: PRIORS

- $M \sim \mathcal{G}(\alpha, \delta)$

- $\beta \sim \mathcal{G}(\rho, \lambda)$

- $p_j \sim \mathcal{B}eta(\mu_j, \sigma_j), \forall j$

Other possibilities *(still to be explored)*

- $p_j = \phi p_{j-1} + \eta$ (provided $0 \leq p_j \leq 1$)
- (A more general) Markov chain for $p_j$'s
- $p_j \sim \mathcal{B}eta(\mu, \sigma), \forall j$

# SELF-EXCITING PROCESS: NOTATIONS

$\underline{p} = (p_1, \ldots, p_n)$

$p_{-j} = (p_1, \ldots, p_{j-1}, p_{j+1}, \ldots, p_n)$

$Z_{-j} = (Z_1, \ldots, Z_{j-1}, Z_{j+1}, \ldots, Z_n)$

From now on, omit dependence on $\underline{t} = (t_1, \ldots, t_n)$

# SELF-EXCITING PROCESS: POSTERIORS

- $M|\beta, Z^{(n)}, \underline{p} \sim \mathcal{G}(\alpha + n, \delta + B(\beta, Z^{(n)}))$

- $\beta|M, Z^{(n)}, \underline{p} \propto \beta^{\rho+n} \prod_{i=1}^{n} A_i(\beta, Z^{(i-1)}) e^{-MB(\beta,Z^{(n)})-\lambda\beta}$

- $p_j|M, \beta, Z^{(n)}, p_{-j} \sim \mathcal{B}eta(\mu_j + Z_j, \sigma_j + (1 - Z_j)), \ \forall j$

# SELF-EXCITING PROCESS: POSTERIORS

$$\mathbb{P}(Z_j = r | M, \beta, \underline{p}, Z_{-j}) = \frac{C_r}{C_0 + C_1}, r = 0, 1$$

$$C_0 = \prod_{i=j+1}^{n} \left[ t_i^{\beta-1} + \sum_{h=1,i-1;h\neq j} Z_h (t_i - t_h)^{\beta} \right]$$

$$C_1 = \prod_{i=j+1}^{n} \left[ t_i^{\beta-1} + \sum_{h=1,i-1;h\neq j} Z_h (t_i - t_h)^{\beta} + (t_i - t_j)^{\beta} \right] e^{-M(T-t_j)^{\beta}}$$

# SELF-EXCITING PROCESS: EXTENSIONS

Different $(M_i, \beta_i) \Rightarrow$ messy computations in general

$M_i = M\rho^i$, $\beta_i = \beta$, $0 < \rho < 1$, $i = 1, \ldots, n$

$\rho \sim \mathcal{B}eta(\phi, \tau)$

- Full conditional posterior for $\rho$, apart from constant

- Same full conditionals for other parameters with changes: $(t_i - t_j) \rightarrow \rho^j(t_i - t_j)$ & $(T - t_j) \rightarrow \rho^j(T - t_j)$

# CHANGE POINTS

F.R. and Sivaganesan (2005)

NHPP with $\lambda(t; M, \beta) = Mg(t, \beta)$, $M, \beta > 0$

- Changes at each failure time
- Changes at a random number of failure times
- Changes at a random number of times

# CHANGE POINTS

Changes at each failure time

- Hierarchical Model
  - $\beta_i$ i.i.d. $\mathcal{LN}(\phi, \sigma^2), i = 0, \ldots, n$
  - $\phi \sim \mathcal{N}(\mu, \tau^2)$
  - $\sigma^2 \sim \mathcal{IG}(\rho, \gamma)$
- Gamma prior for $M$
- Conditional posteriors
  - Gamma for $M$
  - Inverse Gamma for $\sigma^2$
  - Normal for $\phi$
  - Known (apart from a constant) for $\beta_i$'s

  $\Longrightarrow$ Metropolis-Hastings and Gibbs sampling

# CHANGE POINTS

Changes at each failure time

- Dynamic Model
  - $\log \beta_i = \log a + \log \beta_{i-1} + \epsilon_i, i = 1, \ldots, n$
  - $\epsilon_i \sim \mathcal{N}(0, \sigma^2)$
- Priors
  - Gamma for $M$
  - Inverse Gamma for $\sigma^2$
  - Lognormal for $a \mid \sigma^2$
  - Lognormal for $\beta_0 \mid \sigma^2$

# CHANGE POINTS

Changes at each failure time

- Conditional posteriors
  - Gamma for $M$
  - Inverse Gamma for $\sigma^2$
  - Lognormal for $a$
  - Known (apart from a constant) for $\beta_i$'s

  $\Longrightarrow$ Metropolis-Hastings and Gibbs sampling

- Extension $\Rightarrow$ dynamic linear model ($i = 1, \ldots, n$)

  $\log \beta_i = a\theta_i + \epsilon_i$

  $\theta_i = b\theta_{i-1} + \delta_i$

# CHANGE POINTS

Changes at a random number of failures

- Dynamic model as before
- Bernoulli r.v.'s for change/no change
- Beta priors on Bernoulli parameter

Changes at a random number of points
$\Longrightarrow$ Reversible jump MCMC with steps:

- change of $M$ and $\beta$ at a randomly chosen change point
- change to the location of a randomly chosen change point
- "birth" of a new change point at a randomly chosen location in $(0, y]$;
- "death" of a randomly chosen change point

# FUTURE RESEARCH

- Optimal testing policy

- Extensive data analysis (with Tom Mazzuchi)